Exercise 12D: Solutions

1 We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} FP &= RP \\ \sqrt{x^2 + (y-3))^2} &= \sqrt{(y-(-3))^2} \\ x^2 + (y-3)^2 &= (y+3)^2 \\ x^2 + y^2 - 6y + 9 &= y^2 + 6y + 9 \\ x^2 - 12y &= 0 \\ y &= \frac{x^2}{12}. \end{aligned}$$

Therefore, the set of points is a parabola whose equation is $y = \dfrac{x^2}{12}$.

2 We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} FP &= RP \\ \sqrt{x^2 + (y-(-4))^2} &= \sqrt{(y-2)^2} \\ x^2 + (y+4)^2 &= (y-2)^2 \\ x^2 + y^2 + 8y + 16 &= y^2 - 4y + 4 \\ x^2 + 12y &= -12 \\ y &= -\frac{x^2}{12} - 1. \end{aligned}$$

Therefore, the set of points is a parabola whose equation is

$$y = -\frac{x^2}{12} - 1.$$

3 We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} FP &= RP \\ \sqrt{(x-2)^2 + y^2} &= \sqrt{(x-(-4))^2} \\ (x-2)^2 + y^2 &= (x+4)^2 \\ x^2 - 4x + 4 + y^2 &= x^2 + 8x + 16 \\ -12x + y^2 &= 12 \\ x &= \frac{y^2}{12} - 1. \end{aligned}$$

Therefore, the set of points is a (sideways) parabola whose equation is $x = \dfrac{y^2}{12} - 1$.

4 a We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} FP &= RP \\ \sqrt{(x-c)^2 + y^2} &= \sqrt{(x-(-c))^2} \\ (x-c))^2 + y^2 &= (x+c)^2 \\ x^2 - 2cx + c^2 + y^2 &= x^2 + 2cx + c^2 \\ y^2 - 2cx &= +2cx \\ y^2 &= 4cx \\ x &= \frac{y^2}{4c}. \end{aligned}$$

Therefore, the set of points is a (sideways) parabola whose equation is $x = \dfrac{y^2}{4c}$.

b The parabola with equation $x = -\dfrac{y^2}{4c}$ has focus $F(0, c)$ and directrix $x = -c$. For the parabola $x = 3y^2$, we have $\dfrac{1}{4c} = 3$ so that $c = \dfrac{1}{12}$. Therefore, its focus is $(1/12, 0)$ and its directrix is at $x = -\dfrac{1}{12}$.

5 **a** We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} FP &= RP \\ \sqrt{(x-a)^2+(y-b)^2} &= \sqrt{(y-c)^2} \\ (x-a)^2+(y-b)^2 &= (y-c)^2 \\ x^2-2ax+a^2-2by+b^2 &= -2cy+c^2 \\ x^2-2ax+a^2+b^2-c^2 &= 2by-2cy \\ x^2-2ax+a^2+b^2-c^2 &= (2b-2c)y \end{aligned}$$

Solving for y gives, $y = \frac{1}{2b-2c}(x^2-2ax+a^2+b^2-c^2)$.

b Let $a = 1, b = 2$ and $c = 3$ in the above equation. This gives,

$$\begin{aligned} y &= \frac{1}{2b-2c}(x^2-2ax+a^2+b^2-c^2) \\ &= -\frac{1}{2}(x^2-2x-4). \end{aligned}$$

6 Since the parabola has a vertical line of symmetry, its directrix will be a horizontal line, $y = c$. The point $P(7, 9)$ is on the parabola. Therefore, the distance from $P(7, 9)$ to the focus $F(1, 1)$ is the same as the distance from $P(x, y)$ to the line $y = c$. Therefore,

$$\begin{aligned} FP &= RP \\ \sqrt{(7-1)^2+(9-1)^2} &= \sqrt{(9-c)^2} \\ 6^2+8^2 &= (9-c)^2 \\ (9-c)^2 &= 100 \\ 9-c &= \pm 10 \\ c &= 9 \pm 10 \\ &= -1, 19 \end{aligned}$$

Therefore, there are two possibilities for the equation of the directrix: $y = -1$ and $y = 19$.

7 As the focus lies on the line of symmetry, we can suppose that the coordinates of the focus are $(2, a)$. The distance from the focus $(2, a)$ to $P(1, 1)$ is the same as the distance from the line $y = 3$ to the point $P(1, 1)$. Therefore,

$$\begin{aligned} FP &= RP \\ \sqrt{(1-2)^2+(1-a)^2} &= 2 \\ 1+(1-a)^2 &= 4 \\ (1-a)^2 &= 3 \\ 1-a &= \pm\sqrt{3} \\ a &= 1 \pm \sqrt{3} \end{aligned}$$

Therefore. the coordinates of the focus are either $(2, 1+\sqrt{3})$ or $(2, 1-\sqrt{3})$